

# Equilibrated tractions for the Hybrid High-Order method

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## Abstract

We show how to recover equilibrated face tractions for the hybrid high-order method for linear elasticity recently introduced in [1], and prove that these tractions are optimally convergent.

## Résumé

**Tractions équilibrées pour la méthode hybride d'ordre élevé.** Nous montrons comment obtenir des tractions de face équilibrées pour la méthode hybride d'ordre élevé pour l'élasticité linéaire récemment introduite dans [1] et prouvons que ces tractions convergent de manière optimale.

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## 1. Introduction

Let  $\Omega \subset \mathbb{R}^d$ ,  $d \in \{2, 3\}$ , denote a bounded connected polygonal or polyhedral domain. For  $X \subset \overline{\Omega}$ , we denote by  $(\cdot, \cdot)_X$  and  $\|\cdot\|_X$  respectively the standard inner product and norm of  $L^2(X)$ , and a similar notation is used for  $L^2(X)^d$  and  $L^2(X)^{d \times d}$ . For a given external load  $\mathbf{f} \in L^2(\Omega)^d$ , we consider the linear elasticity problem: Find  $\mathbf{u} \in H_0^1(\Omega)^d$  such that

$$2\mu(\nabla_s \mathbf{u}, \nabla_s \mathbf{v})_\Omega + \lambda(\nabla \cdot \mathbf{u}, \nabla \cdot \mathbf{v})_\Omega = (\mathbf{f}, \mathbf{v})_\Omega. \quad (1)$$

with  $\mu > 0$  and  $\lambda \geq 0$  real numbers representing the scalar Lamé coefficients and  $\nabla_s$  denoting the symmetric gradient operator. Classically, the solution to (1) satisfies  $-\nabla \cdot \boldsymbol{\sigma}(\mathbf{u}) = \mathbf{f}$  a.e. in  $\Omega$  with stress tensor  $\boldsymbol{\sigma}(\mathbf{u}) := 2\mu \nabla_s \mathbf{u} + \lambda \mathbf{I}_d(\nabla \cdot \mathbf{u})$ . Denoting by  $T$  an open subset of  $\Omega$  with non-zero Hausdorff measure ( $T$  will represent a mesh element in what follows), partial integration yields the following local equilibrium property:

$$(\boldsymbol{\sigma}(\mathbf{u}), \nabla_s \mathbf{v}_T)_T - (\boldsymbol{\sigma}(\mathbf{u}) \mathbf{n}_T, \mathbf{v}_T)_{\partial T} = (\mathbf{f}, \mathbf{v}_T)_T \quad \forall \mathbf{v}_T \in \mathbb{P}_d^k(T)^d, \quad (2)$$

where  $\partial T$  and  $\mathbf{n}_T$  denote, respectively, the boundary and outward normal to  $T$ . Additionally, the normal interface tractions  $\boldsymbol{\sigma}(\mathbf{u}) \mathbf{n}_T$  are equilibrated across  $\partial T \cap \Omega$ . The goal of this work is to (i) devise a reformulation of the Hybrid High-Order method for linear elasticity introduced in [1] that identifies its local equilibrium properties expressed by a discrete counterpart of (2) and (ii) to show how the corresponding equilibrated face tractions can be obtained by element-wise post-processing. This is an important complement to the original analysis, as local equilibrium is an essential property in practice. The material is organized as follows: in Section 2 we outline the original formulation of the HHO method; in Section 3 we derive the local equilibrium formulation based on a new local displacement reconstruction.

## 2. The Hybrid High-Order method

We consider admissible mesh sequences in the sense of [2, Section 1.4]. Each mesh  $\mathcal{T}_h$  in the sequence is a finite collection  $\{T\}$  of nonempty, disjoint, open, polytopic elements such that  $\overline{\Omega} = \bigcup_{T \in \mathcal{T}_h} \overline{T}$  and

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$h = \max_{T \in \mathcal{T}_h} h_T$  (with  $h_T$  the diameter of  $T$ ), and there is a matching simplicial submesh of  $\mathcal{T}_h$  with locally equivalent mesh size and which is shape-regular in the usual sense. For all  $T \in \mathcal{T}_h$ , the faces of  $T$  are collected in the set  $\mathcal{F}_T$  and, for all  $F \in \mathcal{F}_T$ ,  $\mathbf{n}_{TF}$  is the unit normal to  $F$  pointing out of  $T$ . Additionally, interfaces are collected in the set  $\mathcal{F}_h^i$  and boundary faces in  $\mathcal{F}_h^b$ . The diameter of a face  $F \in \mathcal{F}_h$  is denoted by  $h_F$ . For the sake of brevity, we abbreviate  $a \lesssim b$  the inequality  $a \leq Cb$  for positive real numbers  $a$  and  $b$  and a generic constant  $C$  which can depend on the mesh regularity, on  $\mu$ ,  $d$ , and the polynomial degree, but is independent of  $h$  and  $\lambda$ . We also introduce the notation  $a \simeq b$  for the uniform equivalence  $a \lesssim b \lesssim a$ .

Let a polynomial degree  $k \geq 1$  be fixed. The local and global spaces of degrees of freedom (DOFs) are

$$\underline{\mathbf{U}}_T^k := \mathbb{P}_d^k(T)^d \times \left\{ \bigotimes_{F \in \mathcal{F}_T} \mathbb{P}_{d-1}^k(F)^d \right\} \quad \forall T \in \mathcal{T}_h, \quad \underline{\mathbf{U}}_h^k := \left\{ \bigotimes_{T \in \mathcal{T}_h} \mathbb{P}_d^k(T)^d \right\} \times \left\{ \bigotimes_{F \in \mathcal{F}_h} \mathbb{P}_{d-1}^k(F)^d \right\}. \quad (3)$$

A generic collection of DOFs from  $\underline{\mathbf{U}}_h^k$  is denoted by  $\underline{\mathbf{v}}_h = ((\mathbf{v}_T)_{T \in \mathcal{T}_h}, (\mathbf{v}_F)_{F \in \mathcal{F}_h})$  and, for a given  $T \in \mathcal{T}_h$ ,  $\underline{\mathbf{v}}_T = (\mathbf{v}_T, (\mathbf{v}_F)_{F \in \mathcal{F}_T}) \in \underline{\mathbf{U}}_T^k$  indicates its restriction to  $\underline{\mathbf{U}}_T^k$ . For all  $T \in \mathcal{T}_h$ , we define a high-order local displacement reconstruction operator  $\mathbf{p}_T^k : \underline{\mathbf{U}}_T^k \rightarrow \mathbb{P}_d^{k+1}(T)^d$  by solving the following (well-posed) pure traction problem: For a given  $\underline{\mathbf{v}}_T \in \underline{\mathbf{U}}_T^k$ ,  $\mathbf{p}_T^k \underline{\mathbf{v}}_T$  is such that

$$(\nabla_s \mathbf{p}_T^k \underline{\mathbf{v}}_T, \nabla_s \mathbf{w})_T = (\nabla_s \mathbf{v}_T, \nabla_s \mathbf{w})_T + \sum_{F \in \mathcal{F}_T} (\mathbf{v}_F - \mathbf{v}_T, \nabla_s \mathbf{w} \mathbf{n}_{TF})_F \quad \forall \mathbf{w} \in \mathbb{P}_d^{k+1}(T)^d, \quad (4)$$

and the rigid-body motion components of  $\mathbf{p}_T^k \underline{\mathbf{v}}_T$  are prescribed so that  $\int_T \mathbf{p}_T^k \underline{\mathbf{v}}_T = \int_T \mathbf{v}_T$  and  $\int_T \nabla_{ss}(\mathbf{p}_T^k \underline{\mathbf{v}}_T) = \sum_{F \in \mathcal{F}_T} \int_F \frac{1}{2}(\mathbf{n}_{TF} \otimes \mathbf{v}_F - \mathbf{v}_F \otimes \mathbf{n}_{TF})$  where  $\nabla_{ss}$  is the skew-symmetric gradient operator. Additionally, we define the divergence reconstruction  $D_T^k : \underline{\mathbf{U}}_T^k \rightarrow \mathbb{P}_d^k(T)$  such that, for a given  $\underline{\mathbf{v}}_T \in \underline{\mathbf{U}}_T^k$ ,

$$(D_T^k \underline{\mathbf{v}}_T, q)_T = (\nabla \cdot \mathbf{v}_T, q)_T + \sum_{F \in \mathcal{F}_T} (\mathbf{v}_F - \mathbf{v}_T, q \mathbf{n}_{TF})_F \quad \forall q \in \mathbb{P}_d^k(T). \quad (5)$$

We introduce the local bilinear form  $a_T : \underline{\mathbf{U}}_T^k \times \underline{\mathbf{U}}_T^k \rightarrow \mathbb{R}$  such that

$$a_T(\underline{\mathbf{w}}_T, \underline{\mathbf{v}}_T) := 2\mu \{ (\nabla_s \mathbf{p}_T^k \underline{\mathbf{w}}_T, \nabla_s \mathbf{p}_T^k \underline{\mathbf{v}}_T)_T + s_T(\underline{\mathbf{w}}_T, \underline{\mathbf{v}}_T) \} + \lambda (D_T^k \underline{\mathbf{w}}_T, D_T^k \underline{\mathbf{v}}_T)_T, \quad (6)$$

where the stabilizing bilinear form  $s_T : \underline{\mathbf{U}}_T^k \times \underline{\mathbf{U}}_T^k \rightarrow \mathbb{R}$  is such that

$$s_T(\underline{\mathbf{w}}_T, \underline{\mathbf{v}}_T) := \sum_{F \in \mathcal{F}_T} h_F^{-1} (\pi_F^k(\mathbf{P}_T^k \underline{\mathbf{w}}_T - \mathbf{w}_F), \pi_F^k(\mathbf{P}_T^k \underline{\mathbf{v}}_T - \mathbf{v}_F))_F, \quad (7)$$

and a second displacement reconstruction  $\mathbf{P}_T^k : \underline{\mathbf{U}}_T^k \rightarrow \mathbb{P}_d^{k+1}(T)$  is defined such that, for all  $\underline{\mathbf{v}}_T \in \underline{\mathbf{U}}_T^k$ ,  $\mathbf{P}_T^k \underline{\mathbf{v}}_T := \mathbf{v}_T + (\mathbf{p}_T^k \underline{\mathbf{v}}_T - \pi_T^k \mathbf{p}_T^k)$ . Let  $\underline{\mathbf{I}}_T^k : H^1(T)^d \rightarrow \underline{\mathbf{U}}_T^k$  be the reduction map such that, for all  $T \in \mathcal{T}_h$  and all  $\mathbf{v} \in H^1(T)^d$ ,  $\underline{\mathbf{I}}_T^k \mathbf{v} = (\pi_T^k \mathbf{v}, (\pi_F^k \mathbf{v})_{F \in \mathcal{F}_T})$ . The potential reconstruction  $\mathbf{p}_T^k$  and the bilinear form  $s_T$  are conceived so that they satisfy the following two key properties:

(i) *Stability.* For all  $\underline{\mathbf{v}}_T \in \underline{\mathbf{U}}_T^k$ ,

$$\|\nabla_s \mathbf{p}_T^k \underline{\mathbf{v}}_T\|_T^2 + s_T(\underline{\mathbf{v}}_T, \underline{\mathbf{v}}_T) \simeq \|\nabla_s \mathbf{v}_T\|_T^2 + j_T(\underline{\mathbf{v}}_T, \underline{\mathbf{v}}_T), \quad (8)$$

with bilinear form  $j_T : \underline{\mathbf{U}}_T^k \times \underline{\mathbf{U}}_T^k \rightarrow \mathbb{R}$  such that  $j_T(\underline{\mathbf{w}}_T, \underline{\mathbf{v}}_T) := \sum_{F \in \mathcal{F}_T} h_F^{-1} (\mathbf{w}_T - \mathbf{w}_F, \mathbf{v}_T - \mathbf{v}_F)_F$ .

(ii) *Approximation.* For all  $\mathbf{v} \in H^{k+2}(T)^d$ ,

$$\{\|\nabla_s(\mathbf{v} - \mathbf{p}_T^k \underline{\mathbf{I}}_T^k \mathbf{v})\|_T^2 + s_T(\underline{\mathbf{I}}_T^k \mathbf{v}, \underline{\mathbf{I}}_T^k \mathbf{v})\}^{1/2} \lesssim h_T^{k+1} \|\mathbf{v}\|_{H^{k+2}(T)^d}. \quad (9)$$

We observe that, unlike  $s_T$ , the stabilization bilinear form  $j_T$  only satisfies  $j_T(\underline{\mathbf{I}}_T^k \mathbf{v}, \underline{\mathbf{I}}_T^k \mathbf{v}) \lesssim h^k \|\mathbf{v}\|_{H^{k+1}(T)^d}$ . The discrete problem reads: Find  $\underline{\mathbf{u}}_h \in \underline{\mathbf{U}}_{h,0}^k := \{\underline{\mathbf{u}}_h \in \underline{\mathbf{U}}_h^k \mid \mathbf{u}_F \equiv \mathbf{0} \quad \forall F \in \mathcal{F}_h^b\}$  such that

$$a_h(\underline{\mathbf{u}}_h, \underline{\mathbf{v}}_h) := \sum_{T \in \mathcal{T}_h} a_T(\underline{\mathbf{w}}_T, \underline{\mathbf{v}}_T) = \sum_{T \in \mathcal{T}_h} (\mathbf{f}, \mathbf{v}_T)_T \quad \forall \underline{\mathbf{v}}_h \in \underline{\mathbf{U}}_{h,0}^k. \quad (10)$$

The following convergence result was proved in [1]:

**Theorem 1** (Energy error estimate). *Let  $\mathbf{u} \in H_0^1(\Omega)^d$  and  $\underline{\mathbf{u}}_h \in \underline{\mathbf{U}}_{h,0}^k$  denote the unique solutions to (1) and (10), respectively, and assume  $\mathbf{u} \in H^{k+2}(\Omega)^d$  and  $\nabla \cdot \mathbf{u} \in H^{k+1}(\Omega)$ . Then, letting  $\hat{\underline{\mathbf{u}}}_h \in \underline{\mathbf{U}}_{h,0}^k$  be such that  $\hat{\underline{\mathbf{u}}}_T := \underline{\mathbf{I}}_T^k \mathbf{u}$  for all  $T \in \mathcal{T}_h$ , the following holds (with  $\|\underline{\mathbf{v}}_T\|_{a,T}^2 = a_T(\underline{\mathbf{v}}_T, \underline{\mathbf{v}}_T)$  for all  $\underline{\mathbf{v}}_T \in \underline{\mathbf{U}}_T^k$ ):*

$$\sum_{T \in \mathcal{T}_h} \|\underline{\mathbf{u}}_T - \hat{\underline{\mathbf{u}}}_T\|_{a,T}^2 \lesssim h^{2(k+1)} \left( \|\mathbf{u}\|_{H^{k+2}(\Omega)^d} + \lambda \|\nabla \cdot \mathbf{u}\|_{H^{k+1}(\Omega)} \right)^2. \quad (11)$$

Moreover, assuming elliptic regularity,  $\sum_{T \in \mathcal{T}_h} \|\mathbf{u} - \mathbf{p}_T^k \underline{\mathbf{u}}_T\|_{L^2(T)^d}^2 \lesssim h^{2(k+2)} \left( \|\mathbf{u}\|_{H^{k+2}(\Omega)^d} + \lambda \|\nabla \cdot \mathbf{u}\|_{H^{k+1}(\Omega)} \right)^2$ .

### 3. Local equilibrium formulation

The difficulty in devising an equivalent local equilibrium formulation for problem (10) comes from the stabilization term  $s_T$ , which introduces a non-trivial coupling of interface DOFs inside each element. In this section, we introduce post-processed discrete displacement and stress reconstructions that allow us to circumvent this difficulty. For a given element  $T \in \mathcal{T}_h$ , define the following bilinear form on  $\underline{\mathbf{U}}_T^k$ :

$$\tilde{a}_T(\underline{\mathbf{w}}_T, \underline{\mathbf{v}}_T) := 2\mu \{ (\nabla_s \mathbf{p}_T^k \underline{\mathbf{w}}_T, \nabla_s \mathbf{p}_T^k \underline{\mathbf{v}}_T)_T + j_T(\underline{\mathbf{w}}_T, \underline{\mathbf{v}}_T) \} + \lambda (D_T^k \underline{\mathbf{w}}_T, D_T^k \underline{\mathbf{v}}_T)_T, \quad (12)$$

where the only difference with respect to the bilinear form  $a_T$  defined by (6) is that we have stabilized using  $j_T$  instead of  $s_T$ . We observe that, while proving a discrete local equilibrium relation for the method based on  $\tilde{a}_T$  would not require any local post-processing, the suboptimal consistency properties of  $j_T$  would only yield  $h^{2k}$  in the right-hand side of (11). Denoting by  $\|\cdot\|_{\tilde{a},T}$  the local seminorm induced by  $\tilde{a}_T$  on  $\underline{\mathbf{U}}_T^k$ , one can prove that, for all  $\underline{\mathbf{v}}_T \in \underline{\mathbf{U}}_T^k$ ,

$$\|\underline{\mathbf{v}}_T\|_{\tilde{a},T} \simeq \|\underline{\mathbf{v}}_T\|_{a,T}. \quad (13)$$

We next define the isomorphism  $\underline{\mathbf{c}}_T^k : \underline{\mathbf{U}}_T^k \rightarrow \underline{\mathbf{U}}_T^k$  such that

$$\tilde{a}_T(\underline{\mathbf{c}}_T^k \underline{\mathbf{w}}_T, \underline{\mathbf{v}}_T) = a_T(\underline{\mathbf{w}}_T, \underline{\mathbf{v}}_T) + (2\mu)j_T(\underline{\mathbf{w}}_T, \underline{\mathbf{v}}_T) \quad \forall \underline{\mathbf{v}}_T \in \underline{\mathbf{U}}_T^k, \quad (14)$$

and rigid-body motion components prescribed as above. We also introduce the stress reconstruction  $\mathbf{S}_T^k : \underline{\mathbf{U}}_T^k \rightarrow \mathbb{P}_d^k(T)^{d \times d}$  such that

$$\mathbf{S}_T^k := (2\mu \nabla_s \mathbf{p}_T^k + \lambda \mathbf{I}_d D_T^k) \circ \underline{\mathbf{c}}_T^k. \quad (15)$$

**Lemma 2** (Equilibrium formulation). *The bilinear form  $a_T$  defined by (6) is such that, for all  $\underline{\mathbf{w}}_T, \underline{\mathbf{v}}_T \in \underline{\mathbf{U}}_T^k$ ,*

$$a_T(\underline{\mathbf{w}}_T, \underline{\mathbf{v}}_T) = (\mathbf{S}_T^k \underline{\mathbf{w}}_T, \nabla_s \underline{\mathbf{v}}_T)_T + \sum_{F \in \mathcal{F}_T} (\tau_{TF}(\underline{\mathbf{w}}_T), \mathbf{v}_F - \mathbf{v}_T)_F, \quad (16)$$

with interface traction  $\tau_{TF} : \underline{\mathbf{U}}_T^k \rightarrow \mathbb{P}_{d-1}^k(F)^d$  such that

$$\tau_{TF}(\underline{\mathbf{w}}_T) = \mathbf{S}_T^k \underline{\mathbf{w}}_T \mathbf{n}_{TF} + h_F^{-1} [((\underline{\mathbf{c}}_T^k \underline{\mathbf{w}}_T)_F - \mathbf{w}_F) - ((\underline{\mathbf{c}}_T^k \underline{\mathbf{w}}_T)_T - \mathbf{w}_T)]. \quad (17)$$

*Proof.* Let  $\tilde{\underline{\mathbf{w}}}_T := \underline{\mathbf{c}}_T^k \underline{\mathbf{w}}_T$ . We have, using the definitions (14) of  $\underline{\mathbf{c}}_T^k$  and (12) of the bilinear form  $\tilde{a}_T$ ,

$$\begin{aligned} a_T(\underline{\mathbf{w}}_T, \underline{\mathbf{v}}_T) &= \tilde{a}_T(\tilde{\underline{\mathbf{w}}}_T, \underline{\mathbf{v}}_T) - (2\mu)j_T(\underline{\mathbf{w}}_T, \underline{\mathbf{v}}_T) \\ &= 2\mu \{ (\nabla_s \mathbf{p}_T^k \tilde{\underline{\mathbf{w}}}_T, \nabla_s \mathbf{p}_T^k \underline{\mathbf{v}}_T)_T + j_T(\tilde{\underline{\mathbf{w}}}_T - \underline{\mathbf{w}}_T, \underline{\mathbf{v}}_T) \} + \lambda (D_T^k \tilde{\underline{\mathbf{w}}}_T, D_T^k \underline{\mathbf{v}}_T)_T \\ &= (\mathbf{S}_T^k \underline{\mathbf{w}}_T, \nabla_s \underline{\mathbf{v}}_T)_T + \sum_{F \in \mathcal{F}_T} (\mathbf{S}_T^k \underline{\mathbf{w}}_T \mathbf{n}_{TF}, \mathbf{v}_F - \mathbf{v}_T)_F + (2\mu)j_T(\tilde{\underline{\mathbf{w}}}_T - \underline{\mathbf{w}}_T, \underline{\mathbf{v}}_T), \end{aligned}$$

where we have concluded using (4) with  $\mathbf{w} = \mathbf{p}_T^k \tilde{\underline{\mathbf{w}}}_T$ , (5) with  $q = D_T^k \tilde{\underline{\mathbf{w}}}_T$ , and recalling the definition (15) of  $\mathbf{S}_T^k$ . To obtain (16), it suffices to use the definition of  $j_T$ .  $\square$

**Lemma 3** (Local equilibrium). *Let  $\underline{\mathbf{u}}_h \in \underline{\mathbf{U}}_{h,0}^k$  denote the unique solution to (10). Then, for all  $T \in \mathcal{T}_h$ , the following discrete counterpart of the local equilibrium relation (2) holds:*

$$(\mathbf{S}_T^k \underline{\mathbf{u}}_T, \nabla_s \mathbf{v}_T)_T - \sum_{F \in \mathcal{F}_T} (\boldsymbol{\tau}_{TF}(\underline{\mathbf{u}}_T), \mathbf{v}_T)_F = (\mathbf{f}, \mathbf{v}_T)_T \quad \forall \mathbf{v}_T \in \mathbb{P}_d^k(T)^d, \quad (18)$$

and the numerical flux are equilibrated in the following sense: For all  $F \in \mathcal{F}_h^i$  such that  $F \subset \partial T_1 \cap \partial T_2$ ,

$$\boldsymbol{\tau}_{T_1 F}(\underline{\mathbf{u}}_{T_1}) + \boldsymbol{\tau}_{T_2 F}(\underline{\mathbf{u}}_{T_2}) = \mathbf{0}. \quad (19)$$

*Proof.* To prove (18), let an element  $T \in \mathcal{T}_h$  be fixed, take as an ansatz collection of DOFs in (10)  $\underline{\mathbf{v}}_h = ((\mathbf{v}_T)_{T \in \mathcal{T}_h}, (\mathbf{0})_{F \in \mathcal{F}_h})$  with  $\mathbf{v}_T$  in  $\mathbb{P}_d^k(T)^d$  and  $\mathbf{v}_{T'} \equiv \mathbf{0}$  for all  $T' \in \mathcal{T}_h \setminus \{T\}$ , and use (16) with  $\underline{\mathbf{w}}_T = \underline{\mathbf{u}}_T$  to conclude that  $a_T(\underline{\mathbf{u}}_T, \underline{\mathbf{v}}_T)$  corresponds to the left-hand side of (18). Similarly, to prove (19), let an interface  $F \in \mathcal{F}_h^i$  be fixed and take as an ansatz collection of DOFs in (10)  $\underline{\mathbf{v}}_h = ((\mathbf{0})_{T \in \mathcal{T}_h}, (\mathbf{v}_F)_{F \in \mathcal{F}_h}) \in \underline{\mathbf{U}}_{h,0}^k$  with  $\mathbf{v}_F$  in  $\mathbb{P}_{d-1}^k(F)^d$  and  $\mathbf{v}_{F'} \equiv \mathbf{0}$  for all  $F' \in \mathcal{F}_h \setminus \{F\}$ . Then, using (16) with  $\underline{\mathbf{w}}_T = \underline{\mathbf{u}}_T$  in (10), it is inferred that  $a_h(\underline{\mathbf{u}}_h, \underline{\mathbf{v}}_h) = (\boldsymbol{\tau}_{T_1 F}(\underline{\mathbf{u}}_{T_1}) + \boldsymbol{\tau}_{T_2 F}(\underline{\mathbf{u}}_{T_2}), \mathbf{v}_F)_F = 0$ , which proves the desired result since  $\boldsymbol{\tau}_{T_1 F}(\underline{\mathbf{u}}_{T_1}) + \boldsymbol{\tau}_{T_2 F}(\underline{\mathbf{u}}_{T_2}) \in \mathbb{P}_{d-1}^k(F)^d$ .  $\square$

To conclude, we show that the locally post-processed solution yields a new collection of DOFs that is an equally good approximation of the exact solution as is the discrete solution  $\underline{\mathbf{u}}_h$ . Consequently, the equilibrated face numerical tractions defined in (17) optimally converge to the exact tractions.

**Proposition 4** (Convergence for  $\underline{\mathbf{c}}_T^k \underline{\mathbf{u}}_T$ ). *Using the notation of Theorem 1, the following holds:*

$$\sum_{T \in \mathcal{T}_h} \|\underline{\mathbf{c}}_T^k \underline{\mathbf{u}}_T - \hat{\underline{\mathbf{u}}}_T\|_{a,T}^2 \lesssim h^{2(k+1)} (\|\mathbf{u}\|_{H^{k+2}(\Omega)^d} + \lambda \|\nabla \cdot \mathbf{u}\|_{H^{k+1}(\Omega)})^2. \quad (20)$$

*Proof.* Let  $T \in \mathcal{T}_h$ . Recalling (14), we have

$$\begin{aligned} \tilde{a}_T(\underline{\mathbf{c}}_T^k \underline{\mathbf{u}}_T - \hat{\underline{\mathbf{u}}}_T, \underline{\mathbf{v}}_T) &= a_T(\underline{\mathbf{u}}_T, \underline{\mathbf{v}}_T) + (2\mu)j_T(\underline{\mathbf{u}}_T, \underline{\mathbf{v}}_T) - \tilde{a}_T(\hat{\underline{\mathbf{u}}}_T, \underline{\mathbf{v}}_T) \\ &= a_T(\underline{\mathbf{u}}_T - \hat{\underline{\mathbf{u}}}_T, \underline{\mathbf{v}}_T) + (2\mu)s_T(\hat{\underline{\mathbf{u}}}_T, \underline{\mathbf{v}}_T) + (2\mu)j_T(\underline{\mathbf{u}}_T - \hat{\underline{\mathbf{u}}}_T, \underline{\mathbf{v}}_T). \end{aligned}$$

Hence, using the Cauchy–Schwarz inequality followed by the stability property (8) and multiple applications of the norm equivalence (13),

$$\begin{aligned} |\tilde{a}_T(\underline{\mathbf{c}}_T^k \underline{\mathbf{u}}_T - \hat{\underline{\mathbf{u}}}_T, \underline{\mathbf{v}}_T)| &\leq \left\{ \|\underline{\mathbf{u}}_T - \hat{\underline{\mathbf{u}}}_T\|_{a,T}^2 + (2\mu)s_T(\hat{\underline{\mathbf{u}}}_T, \hat{\underline{\mathbf{u}}}_T) + (2\mu)j_T(\underline{\mathbf{u}}_T - \hat{\underline{\mathbf{u}}}_T, \underline{\mathbf{u}}_T - \hat{\underline{\mathbf{u}}}_T) \right\}^{1/2} \|\underline{\mathbf{v}}_T\|_{\tilde{a},T} \\ &\lesssim \left\{ \|\underline{\mathbf{u}}_T - \hat{\underline{\mathbf{u}}}_T\|_{a,T}^2 + (2\mu)s_T(\hat{\underline{\mathbf{u}}}_T, \hat{\underline{\mathbf{u}}}_T) \right\}^{1/2} \|\underline{\mathbf{v}}_T\|_{\tilde{a},T}. \end{aligned}$$

Using again (13) followed by the latter inequality, we infer that

$$\|\underline{\mathbf{c}}_T^k \underline{\mathbf{u}}_T - \hat{\underline{\mathbf{u}}}_T\|_{a,T} \lesssim \|\underline{\mathbf{c}}_T^k \underline{\mathbf{u}}_T - \hat{\underline{\mathbf{u}}}_T\|_{\tilde{a},T} = \sup_{\underline{\mathbf{v}}_T \in \underline{\mathbf{U}}_T^k \setminus \{\mathbf{0}\}} \frac{\tilde{a}_T(\underline{\mathbf{c}}_T^k \underline{\mathbf{u}}_T - \hat{\underline{\mathbf{u}}}_T, \underline{\mathbf{v}}_T)}{\|\underline{\mathbf{v}}_T\|_{\tilde{a},T}} \lesssim \left\{ \|\underline{\mathbf{u}}_T - \hat{\underline{\mathbf{u}}}_T\|_{a,T}^2 + (2\mu)s_T(\hat{\underline{\mathbf{u}}}_T, \hat{\underline{\mathbf{u}}}_T) \right\}^{1/2}.$$

The estimate (20) then follows squaring the above inequality, summing over  $T \in \mathcal{T}_h$ , and using (11) and (9), respectively, to bound the terms in the right-hand side.  $\square$

To assess the estimate (20), we have numerically solved the pure displacement problem with exact solution  $\mathbf{u} = (\sin(\pi x_1) \sin(\pi x_2) + 1/2 x_1, \cos(\pi x_1) \cos(\pi x_2) + 1/2 x_2)$  for  $\mu = \lambda = 1$  on a  $h$ -refined sequence of triangular meshes. The corresponding convergence results are presented in Figure 1. In the left panel, we compare the quantities on the left-hand side of estimates (11) and (20). Although the order of convergence is the same, the original solution  $\underline{\mathbf{u}}_h$  displays better accuracy in the energy-norm. This is essentially due to face unknowns, as confirmed in the right panel, where the square roots of the quantities  $\sum_{T \in \mathcal{T}_h} \|\underline{\mathbf{u}}_T - \hat{\underline{\mathbf{u}}}_T\|_T^2$  and  $\sum_{T \in \mathcal{T}_h} \|\underline{\mathbf{c}}_T^k \underline{\mathbf{u}}_T - \hat{\underline{\mathbf{u}}}_T\|_T^2$  (both of which are discrete  $L^2$ -norms of the error) are plotted.

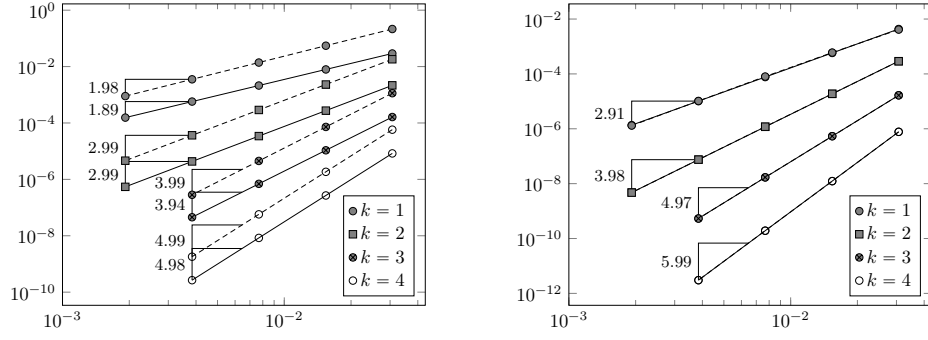


Figure 1: Convergence results in the energy-norm (left) and  $L^2$ -norm (right) for the solution to (10) (solid lines) and its post-processing based on  $\mathcal{C}_T^k$  (dashed lines). The right panel shows that the post-processing has no sizable effect on element unknowns.

## References

- [1] D. A. Di Pietro, A. Ern, A hybrid high-order locking-free method for linear elasticity on general meshes, *Comput. Meth. Appl. Mech. Engrg.* 283 (2015) 1–21.
- [2] D. A. Di Pietro, A. Ern, *Mathematical aspects of discontinuous Galerkin methods*, Vol. 69 of *Mathématiques & Applications*, Springer-Verlag, Berlin, 2012.